Fracture Network Response to Injection
Using an Efficient Displacement Discontinuity Method

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**ABSTRACT**

Unconventional geothermal reservoirs play an important role in supplying fuel for a growing energy demand in the U.S. Development of such reservoirs relies on creating a fracture network to provide flow and transport conduits. The fractures represent the main channels for the movement of the underground fluid during injection and production operations. Geomechanical interactions among all fractures (natural and hydraulically induced one) change the initial stress distribution of the rock and then impacts fracture initialization, and propagation processes. As the later governs the flow pattern and total heat and fluid recovery in the reservoir, having such stress variations at high-resolution level could be useful for a realistic representation and geomechanical characterization of those large-scale unconventional reservoirs, as well as, for instance, during a massive fracture prediction analysis, and the proper design of exploitation strategies.

The displacement discontinuity method (DDM) is frequently used for modeling the behavior of fractures embedded in linear-elastic rocks. However, DDM is not computational efficient for very large systems of cracks, often limiting its application to small-scale problems. Fast summation techniques such as the Fast Multipole Method (FMM) can speed up the analysis of fracture problems, even using personal computers with modest computational resources. This work presents a novel approach for the efficient post-processing computation of regional stress variations in large-scale fractured reservoirs combining DDM and FMM. Several case studies involving naturally fractured reservoirs are treated and the effect of several geomechanical aspects such as non-linear joint deformation and fracture fluid pressure, are evaluated. Results of the approach show a good agreement with conventional solutions, demonstrating its efficiency and accuracy for large-scale situations, as well as its usefulness for the geomechanical characterization of unconventional reservoirs.

1. Introduction

Unconventional geothermal reservoirs can play an important role in supplying fuel for a growing energy demand in the U.S. Development of such reservoirs relies on developing a fracture network to provide flow and transport conduits. The fractures represent the main channels for the movement of the underground fluid during injection and production operations. Geomechanical interactions among all fractures (natural and hydraulically induced one) change the initial stress distribution of the rock and then impacts fracture initialization, and propagation processes. As the later governs the flow pattern and total fluid recovery in the reservoir, having such stress variations at high-resolution level could be useful for a realistic representation and geomechanical characterization of those large-scale unconventional reservoirs, as well as, for instance, during a massive fracture prediction analysis, and the proper design of exploitation strategies.

An approach to modeling fracture reservoirs uses 3D FEM along with a stochastic network of fractures [1]. This approach, however, is somewhat limited when it comes to accounting for mechanical interactions among the fractures within the network. An alternative is to use a Displacement Discontinuity Method [2-5]. However, DDM requires computing the influences among all elements so the coefficient matrix of the system of equations is dense and nonsymmetrical. This requirement makes matrix-vector products time-consuming not only for the solution of linear system of equations but also for post-processing computation when large numbers of fracture elements and field points locations are involved, making DDM computationally intensive. However, the Fast Multipole Method (FMM) [5] has provided a remedy. The FMM relies on a strategy in which the matrix-vector multiplication is accelerated without forming the coefficient matrix explicitly. In recent years, a new FMM formulation called “black-box” has been developed which does not requires the implementation of multiple expansions of the underlying kernel but only the kernel evaluations [6].
Previous works using DDM and FMM has been limited [7, 8]. Both studies used the available mathematical developments of FMM but exclusively for the solution of system linear of equations. On the other hand, conventional FMM formulations has been also applied to speed up visualization operations in electric [9], magnetic [10], electromagnetic [11, 12] and acoustic [13] problems. However, only few FMM applications have been reported for the fast evaluation, as post-processing stage, of internal stress of some elastic structures [14] and plates with internal holes and small cracks [14-16].

This work uses both the DDM and the black-box FMM formulations (acronymed as DFMM) for a rapid stress computation of fractures problems in elastic rocks via the acceleration of matrix-vector products. Using fundamental solutions for constant small cracks [14-16].

2. Mathematical Formulations

2.1. Displacement Discontinuity Method (DDM)

DDM is an indirect Boundary Element Method (BEM) for modeling the normal (opening) and shear (slip) displacement discontinuities of fractures embedded in an infinite and elastic medium [17]. The method is based on fundamental or analytical solutions to the problem of a finite segment fracture centered in the domain caused by displacement discontinuities (\(D_s\) and \(D_n\)) of the fracture with its center at the origin (source) [17]:

\[
\begin{align*}
\sigma_{xx} &= 2GD_n \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) + 2GD_s \left( \frac{\partial^3 F}{\partial x \partial y} + \frac{\partial^3 F}{\partial x \partial y^2} \right) \\
\sigma_{yy} &= 2GD_n \left( \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2} \right) - 2GD_s \left( \frac{\partial^3 F}{\partial x \partial y^2} \right) \\
\sigma_{xy} &= -2GD_s \left( \frac{\partial^3 F}{\partial x \partial y^2} \right) + 2GD_s \left( \frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y \partial y^2} \right)
\end{align*}
\]

where

\[
F(x, y) = \frac{1}{4\pi(1-v)} \left[ \ln \frac{y}{x-a} - \ln \frac{y}{x+a} \right] - (x-a) \ln \left( \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \right)
\]

In Eqs. (2) and (3), \(F\) is the relative position function between the source \((\bar{x}, \bar{y})\) and field (origin) points, \(G\) is the shear modulus, \(v\) represents the Poisson’s ratio of the solid medium, and \(a\) the fracture half-length.

Using the superposition, the induced stresses at a field point \(i\) caused by the effect of \(m\) fractures at different source locations can be computed as the sum of the contributions of the \(m\) fracture segment involved. After rotating the stresses components to the normal (\(\sigma_n\)) and shear directions (\(\sigma_s\)), we have:

\[
\sigma_n^i = \sum_{j=1}^{m} \sigma_n^{ij} = \sum_{j=1}^{m} A_{nn} D_n + \sum_{j=1}^{m} A_{ns} D_s
\]

\[
\sigma_s^i = \sum_{j=1}^{m} \sigma_s^{ij} = \sum_{j=1}^{m} A_{sn} D_n + \sum_{j=1}^{m} A_{ss} D_s
\]

with

\[
\sigma_n^{ij} = (\sigma_{yy} - \sigma_{xx}) \sin \beta_j \cos \beta_j + \sigma_{xy} \left( \cos \beta_j^2 - \sin \beta_j^2 \right)
\]

\[
\sigma_s^{ij} = \sigma_{xx} \sin \beta_j^2 - 2\sigma_{xy} \sin \beta_j \cos \beta_j + \sigma_{yy} \cos \beta_j^2
\]

and

\[
\sigma_s^{ij} = \sigma_n^{ij} - P
\]

where \(\sigma_n^{ij}\) is the effective normal stress, \(P\) is the fluid pressure, and \(\sigma_{xx}, \sigma_{yy}, \text{ and } \sigma_{xy}\) are the field stresses components in \(xx, yy, \text{ and } xy\) directions, respectively. Note that the coefficients \(A_{nn}, A_{ns}, A_{sn}, \text{ and } A_{ss}\) represent the global influence containing multiple spatial derivative of \(F\) and depending of the relative position of the element \(i\) and \(j\) and upon the orientation and length of fracture element \(j\).

If the mechanical response of a natural fracture (joint) is taken into account [4], then the total displacement discontinuities are solved simply by adding a local or self-influence coefficient to Eq. (4) as:

\[
\begin{align*}
\sigma_n^{ij}_s &= K_n D_s + \sum_{j=1}^{m} A_{nn} D_n + \sum_{j=1}^{m} A_{ns} D_s \\
\sigma_s^{ij}_n &= K_s D_s + \sum_{j=1}^{m} A_{sn} D_n + \sum_{j=1}^{m} A_{ss} D_s
\end{align*}
\]
where \((\sigma_n)^{\alpha_n}\) is the initial stress field, \(K_n\) and \(K_p\) represent the normal and shear joint stiffness. Based on the Goodman model [18], \(K_n\) is estimated with an hyperbolic equation as a function of the initial stiffness \((K_{m})\) and maximum closure \((D_{max})\):

\[
K_n = \frac{K_m}{\left(1 - \frac{\sigma_n'}{K_mD_{max}} + \sigma_n''\right)^2}
\]  

(9)

In a problem with multiple field and source points, a linear system of equations can be formed as:

\[
[A]{x} = \{b\}
\]  

(10)

where \(A\) is the coefficient matrix, \(b\) is a vector of stresses, and \(x\) represents the shear and normal displacement discontinuities. Depending on the specification of boundary conditions, Eq. (10) can be used to: (a) obtain the field stress after a single matrix-vector multiplication if \(x\) is given or, (b) compute \(x\) if \(b\) is specified after solving the system of equations, in which several matrix-vector products could be involved if iterative methods are applied. In this work, we are interested to evaluate the first scenario assuming a numerical solution is given for a particular problem of interest. Note that DDM requires \(O(NM)\) floating point operations to compute and store the geomechanical interactions between the \(N\) and \(M\) degree of freedom associated with the source and field points, respectively. Unfortunately, straightforward direct matrix-vector product is an inefficient way when facing large problem sizes (>10^5 DOFs) which are still beyond the current capability of common personal computers. It requires \(O(N^2)\) memory and operations to compute and stored the coefficient matrix. FMM can be used to accelerate the solutions of DDM by reducing both memory and CPU time to a factor proportional to \(O(N)\).

### 2.2. Fast Multipole Method (FMM)

#### 2.2.1. FMM philosophy

The main idea behind the FMM is to accelerate matrix–vector products \((Ax)\) in iterative algorithms without forming the coefficient matrix explicitly, reducing then computation time and saving memory [19,20]. Algebraically, the product of the \(i\)-th row of a \(N\times N\) matrix \(K\) with a column vector \(\sigma\) of length \(N\) can be expressed as follow:

\[
f(x) = \sum_{i=1}^{N} K(x,y)\sigma_j
\]  

(11)

In analogy to Eq. (8), Eq. (11) may represent a field value \(f\) evaluated at point \(x\) due to the influence of sources (governed by the kernel matrix \(K\)) and located at a set of centers. This computation represents the well-known N-body problem and requires \(N\times N\) algebraic products or \(O(N^2)\) operations to compute the field values at the set of \(N\) evaluation points. \[9\]. Therefore, the objective of using FMM is to reduce this calculation to ideally \(O(N)\) counts by approximating the values of \(f\). This approximation is accomplished by classifying the influences into near and far-field interactions, according with the distance among field and source points. While the near field interactions are evaluated as the conventional DDM showing a quadratic complexity, the far-field influences that involves most of the algebraic products are calculated efficiently using FMM to reduce the cost proportional to \(N, O(N)\).

The far-field interactions is approximated by concentrating the influence of a cluster of source particles (or fractures in our case) in a single location, assuming that the influences of such particles become weaker as the distance between field and source locations increases. This procedure requires to construct a hierarchical tree (or quadtree structure in 2D) to decompose the computational domain and subdivide it at increasing level of refinements, identifying a near and far sub-domains for each level. Based on this space decomposition, Eq. (11) can be expressed as [7, 8, 20]:

\[
f(x) = \sum_{l=1}^{N_{near}} K(x, \gamma)\sigma_j + \sum_{k=1}^{N_{far}} K(x, \gamma)\sigma_k
\]  

(12)

where \(N_{near}\) and \(N_{far}\) represent the number of source points in the near and far-field domains of the summation terms (assuming \(N_{near} << N_{far}\)) which are computed using the conventional DDM and FMM, respectively. More details and variations of the FMM have been described previously by other authors [19, 21].

![Figure 2. Principal operations of the FMM. [22]](image)
common formation properties and FMM parameters used. Constant and variable fluid pressure inside the fractures was evaluated, assuming homogeneous and isotropic conditions. For all cases studies, DDFMM is used to compute in a fast and accurate way the field stresses in the domain, with user-prescribed accuracy, once analytical and numerical solutions are provided.

Table 1. Common data set used in the numerical examples.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain length, ( L ) (m)</td>
<td>200, 400</td>
</tr>
<tr>
<td>Shear modulus, ( G ) (GPa)</td>
<td>15</td>
</tr>
<tr>
<td>Poisson’s ratio, ( v ) (-)</td>
<td>0.1</td>
</tr>
<tr>
<td>Initial normal stiffness, ( K_n ) (GPa/m)</td>
<td>30</td>
</tr>
<tr>
<td>Shear joint stiffness, ( K_s ) (GPa/m)</td>
<td>15</td>
</tr>
<tr>
<td>Initial fracture aperture, ( D_0 ) (m)</td>
<td>3x10^{-3}</td>
</tr>
<tr>
<td>Maximum fracture closure, ( D_{max} ) (m)</td>
<td>3x10^{-3}</td>
</tr>
<tr>
<td>Initial field stress in ( xx ) direction, ( \sigma_{xx} ) (MPa)</td>
<td>20</td>
</tr>
<tr>
<td>Initial field stress in ( yy ) direction, ( \sigma_{yy} ) (MPa)</td>
<td>20</td>
</tr>
<tr>
<td>Initial field stress in ( xy ) direction, ( \sigma_{xy} ) (MPa)</td>
<td>0</td>
</tr>
<tr>
<td>Fluid pressure, ( p ) (MPa)</td>
<td>15</td>
</tr>
<tr>
<td>Number of chebyshev nodes per dimension, ( n ) (-)</td>
<td>5</td>
</tr>
</tbody>
</table>

4.1. Case 1: Model Verification

DDFMM was verified comparing its predictions with an analytical solution for a single crack problem. As shown in Fig. 3, the situation of interest involves a horizontal pressurized line fracture of length \( 2b \) embedded in an infinite 2-D elastic rock. The fracture is centered at the origin of the coordinate system and oriented along the \( x \)-direction.

An analytical approximation for the normal displacement discontinuity (\( D_n \)) is available from the elastic theory as follow [17]:

\[
D_n(x) = -\frac{(1-v)}{G}p\sqrt{b^2-x^2}
\]

(13)

In the numerical analysis, our computational model divided the fracture using 1000 constant length elements and the fracture opening at every segment was solved and compared with analytical values (See Fig. 4).

From Fig. 4 it is clear that the numerical results agree well with the analytical solution when increasing the number of elements.

For the finest discretization, the maximum relative error is less than 0.03%. One satisfied this initial verification stage, DDFMM was then used to compute high quality normal stress distributions of well-known fracture problems (See Figs. 5 and 6). For both cases, 1,000 source elements with 10,000 field points were employed for the visualization whose results agree with those provided by widely used numerical tools (TWODD) [17] and reported in the literature [23, 24].

4.2. Case 2: Computational Performance Evaluation

This case study evaluates the execution time of DDFMM for visualizing large-scale situations. Fig. 7 shows the CPU time used by DDM and DDFMM to compute the normal stress distribution around 10,000 horizontal fractures with increasing number of field points. Table 2 presents the associated numerical values of Fig. 7.

![Figure 3](image3.png)

Figure 3. A pressurized single crack in an infinite domain configuration used to verify DDFMM – Case 1.

![Figure 4](image4.png)

Figure 4. Comparison of the fracture opening between DDFMM and the analytical solution of the pressurized crack problem for different discretizations – Case 1.

![Figure 5](image5.png)

Figure 5. High quality normal stress distribution around a pressurized fracture computed with FMM. 1,000 source elements with 10,000 field points was used for the visualization – Case 1(a).
Note in this figure how the complexity of the matrix-vector multiplication required to compute the stress components at the prescribed field locations has been successfully reduced from a quadratic calculation with conventional DDM to linear one using DDFMM. According with the values in Table 2, an average speed-up, which is the ratio between the execution time of DDM and DDFMM, of 4.5 was gained over the conventional DDM for this particular case study, with an apparent superior performance for higher number of fractures.

**Table 2.** Computational performance of DDFMM and DDM for different visualization data-sets – Case 2.

<table>
<thead>
<tr>
<th>Filed points</th>
<th>Execution time [sec]</th>
<th>Speed up [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDM</td>
<td>DDFMM</td>
<td></td>
</tr>
<tr>
<td>100,000</td>
<td>335</td>
<td>69</td>
</tr>
<tr>
<td>200,000</td>
<td>661</td>
<td>165</td>
</tr>
<tr>
<td>400,000</td>
<td>1328</td>
<td>261</td>
</tr>
<tr>
<td>600,000</td>
<td>2069</td>
<td>481</td>
</tr>
<tr>
<td>1,000,000</td>
<td>-</td>
<td>645</td>
</tr>
</tbody>
</table>

Fig. 8 shows the normal stress distribution for the fracture network with the highest number of field points (one million) using DDFMM. Additionally, Fig. 9 presents a comparison of the normal stress distribution within the small zone of Figure 8 computed with DDM (left) and FMM (right) using 200,000 and one million field points, respectively.
above spatial distribution computed with DDFMM (right) within a small domain with those obtained with DDM (left) using fewer number of field points (200,000). Note how by the inclusion of more field locations additional and subtle details are captured for a better representation (more similar to that presented in Fig. 6 for parallel fractures) with a comparable computation time.

4.3. Case 3: Geomechanical Response of Natural Fractured Reservoirs Due to Injection and Production Operations

This case study computes the stress distributions in a reservoir at different times caused by the isothermal fluid injection through an irregular fracture network. The dimension and distribution of the 1000 fractures (16.6 meters each one) as well as the locations of the injector and producer wells at the center of the fracture elements are showed in Fig. 10. A large number of field points (160,000) were used to capture higher details of the stress variations above and below the initial field stress. Transient numerical solutions was provided via direct methods by solving Eq. (10) for the stress components coupled with the mass balance equation in the fracture network following [3, 4] to accounting for pressure changes, neglecting fluid flow from the fracture to the formation (leak-off).

Fig 12 presents the mean stress distribution in the reservoir using 160,000 field points after 2 (left) and 60 (right) days of injection. The small dashed zone on the bottom left is used in Fig. 13 to compare, with more details, stress variations near the injector well. A quick examination of Fig. 12 shows that slight changes of the initial stress occurred at the beginning of the injection and increase in magnitude during the process. Note that at later times, higher values of values are concentrated around the intersection of fractures where most of them seem to be distributed along NE-SW directions. This orientation coincides with the natural path for the movement of fluids from the injector to the producer wells where greater normal displacements (and then normal stress) are created along this direction to provide transport conduits to the fluid flow. Finally as can be seem in Fig. 13 (right), a closed inspection of the stress variation near the injector reveals how larger number of field points help to construct a more accurate but computationally cheap spatial distribution in this small area using DDFMM, showing again higher stress concentrations at the fracture’s intersections that could potentially lead to, for instance, fracture initiation and propagation.

5. Conclusions

This paper presents a novel approach for the efficient computation of fracture network response to isothermal injection and the resulting variation in the stress field within subsurface reservoirs. The approach called DDFMM combines DDM and FMM to compute large-scale visualization data sets with modest computational resources to obtain high-resolution stress distributions.
as a post-processing stage once the displacement discontinuities on the fractures are provided. DDFMM showed higher computational performance than conventional DDM based on direct matrix-vector multiplications, reducing the execution time to a linear complexity and demonstrating its potential usefulness for realistic representation and geomechanical characterization of large-scale unconventional reservoirs.

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